

# Analogues for Bessel Functions of the Christoffel-Darboux Identity

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## Abstract

We derive analogues for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials:

$$\sum_{k=1}^{\infty} 2(\nu + k) J_{\nu+k}(w) J_{\nu+k}(z) = \frac{wz}{w-z} \left( J_{\nu+1}(w) J_{\nu}(z) - J_{\nu}(w) J_{\nu+1}(z) \right)$$

and

$$2(\nu + 1) \sum_{k=1}^{\infty} 2(\nu + 2k) J_{\nu+2k}(w) J_{\nu+2k}(z) = \frac{w^2 z^2}{w^2 - z^2} \left( J_{\nu+2}(w) J_{\nu}(z) - J_{\nu}(w) J_{\nu+2}(z) \right)$$

for any distinct nonzero complex variables  $w$  and  $z$ , and any complex number  $\nu$ , where  $J_{\mu}$  is the Bessel function of the first kind of order  $\mu$ , for any complex number  $\mu$ . We also provide certain straightforward consequences of these identities.

## 1 Introduction

Via some brief manipulations, the present note derives fairly simple expressions for

$$\sum_{k=1}^{\infty} 2(\nu + k) J_{\nu+k}(w) J_{\nu+k}(z) \tag{1}$$

and

$$\sum_{k=1}^{\infty} 2(\nu + 2k) J_{\nu+2k}(w) J_{\nu+2k}(z) \tag{2}$$

for any distinct nonzero complex variables  $w$  and  $z$ , where  $\nu$  is a complex number, and  $J_{\mu}$  is the Bessel function of the first kind of order  $\mu$ , for any complex number  $\mu$  (see, for example, [9]).

The simplified expressions for (1) and (2) are analogues for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials (see, for example, [5]). See [8] for an application of the results of the present note.

A number of mathematicians have published simplified expressions for the series (1) and (2); the initial publications include [1], [4], [10], and the 1937 edition of [6]. See Section 16.32 of [9] for a description of Kapteyn's and Watson's researches, and Section 11.13 of [6] for a description of Titchmarsh's researches. See also [10] and [11] for a comprehensive account of the literature concerning (1), (2), including Bateman's, Kapteyn's, Titchmarsh's, Watson's, and Wilkins' results, and their relevance in the theory of what are known as Neumann series. See [2] for a treatment of

certain related series that is somewhat similar to that of the present note. See [7] for analogues for Bessel functions of the Christoffel-Darboux identity that are based on differential equations; the present note relies on recurrence relations (that is to say, difference equations).

The present note has the following structure: Section 2 summarizes several classical facts about Bessel functions, Section 3 provides proof of a rather trivial fact about Bessel functions, Section 4 derives simplified expressions for (1) and (2) directly from the facts in Sections 2 and 3, and Section 5 generalizes the results of Section 4.

## 2 Preliminaries

This section provides several well-known facts about Bessel functions. All of these facts can be found, for example, in [9].

For any complex number  $\nu$ , we define  $J_\nu$  to be the (generally multiply valued) Bessel function of the first kind of order  $\nu$  (see, for example, [9]).

The following theorem states a basic symmetry of a Bessel function.

**Theorem 1** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$J_\nu(z e^{\pi i}) = e^{\pi i \nu} J_\nu(z) \quad (3)$$

*for any nonzero complex variable  $z$ .*

**Proof.** Formula 1 of Section 3.62 in [9] provides a slightly more general formulation of (3).  $\square$

The following theorem states that two Bessel functions are orthogonal on  $(0, \infty)$  with respect to the weight function  $w(x) = \frac{1}{x}$  when their orders differ by a nonzero even integer.

**Theorem 2** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\int_0^\infty dx \frac{1}{x} J_{\nu+2j}(x) J_{\nu+2k}(x) = \begin{cases} 0, & j \neq k \\ \frac{1}{2(\nu+2j)}, & j = k \end{cases} \quad (4)$$

*for any integers  $j$  and  $k$  such that the real part of  $\nu + j + k$  is positive.*

**Proof.** Formula 7 of Section 13.41 in [9] provides a slightly more general formulation of (4).  $\square$

The following theorem provides what is known as the Poisson integral representation of a Bessel function.

**Theorem 3** *Suppose that  $\nu$  is a complex number such that the real part of  $\nu$  is greater than  $-\frac{1}{2}$ .*

*Then,*

$$J_\nu(z) = \frac{z^\nu}{\sqrt{\pi} 2^\nu \Gamma(\nu + 1/2)} \int_{-1}^1 du e^{iuz} (1 - u^2)^{\nu-1/2} \quad (5)$$

*for any complex variable  $z$ , where  $\Gamma$  is the gamma (factorial) function.*

**Proof.** Formula 4 of Section 3.3 in [9] provides an equivalent formulation of (5).  $\square$

The following three theorems provide recurrence relations for Bessel functions and their derivatives.

**Theorem 4** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\frac{d}{dz} J_{\nu+1}(z) = J_{\nu}(z) - \frac{\nu+1}{z} J_{\nu+1}(z) \quad (6)$$

*for any nonzero complex variable  $z$ .*

**Proof.** Formula 3 of Section 3.2 in [9] provides an equivalent formulation of (6).  $\square$

**Theorem 5** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\frac{d}{dz} J_{\nu}(z) = \frac{\nu}{z} J_{\nu}(z) - J_{\nu+1}(z) \quad (7)$$

*for any nonzero complex variable  $z$ .*

**Proof.** Formula 4 of Section 3.2 in [9] provides an equivalent formulation of (7).  $\square$

**Theorem 6** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\frac{2\nu}{z} J_{\nu}(z) = J_{\nu-1}(z) + J_{\nu+1}(z) \quad (8)$$

*for any nonzero complex variable  $z$ .*

**Proof.** Formula 1 of Section 3.2 in [9] provides an equivalent formulation of (8).  $\square$

The following corollary is an immediate consequence of (8).

**Corollary 7** *Suppose that  $\nu$  is a complex number such that  $\nu \neq 1$  and  $\nu \neq -1$ .*

*Then,*

$$\frac{2\nu}{z^2} J_{\nu}(z) = \frac{1}{2(\nu-1)} J_{\nu-2}(z) + \frac{\nu}{\nu^2-1} J_{\nu}(z) + \frac{1}{2(\nu+1)} J_{\nu+2}(z) \quad (9)$$

*for any nonzero complex variable  $z$ .*

### 3 A simple technical fact

The purpose of this section is to provide proof of Theorem 9, stating the unsurprising and entirely trivial fact that, for any nonzero complex variable  $z$ ,  $J_{\nu+k}(z)$  tends to 0 as  $k$  tends toward  $\infty$  through the integers.

The following theorem bounds the absolute value of the gamma (factorial) function evaluated at a certain complex number.

**Theorem 8** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\left| \Gamma\left(\nu + k + \frac{1}{2}\right) \right| \geq (k-n)! \left| \Gamma\left(\nu + n + \frac{1}{2}\right) \right| \quad (10)$$

*for any integers  $k$  and  $n$  such that  $k > n > |\nu| + \frac{1}{2}$ , where  $\Gamma$  is the gamma (factorial) function.*

**Proof.** We have that

$$\left| \Gamma\left(\nu + k + \frac{1}{2}\right) \right| = \left| \Gamma\left(\nu + n + \frac{1}{2}\right) \right| \prod_{j=1}^{k-n} \left| \nu + k - j + \frac{1}{2} \right| \quad (11)$$

for any integers  $k$  and  $n$  such that  $k > n > |\nu| + \frac{1}{2}$ . Furthermore,

$$\left| \nu + k - j + \frac{1}{2} \right| \geq k - n - j + 1 \quad (12)$$

for any integers  $j$ ,  $k$ , and  $n$  such that  $k > n > |\nu| + \frac{1}{2}$  and  $j \leq k - n$ . Combining (11) and (12) yields (10).  $\square$

The following technical theorem is the principal purpose of this section.

**Theorem 9** *Suppose that  $\nu$  is a complex number.*

*Then, for any nonzero complex variable  $z$ ,  $J_{\nu+k}(z)$  tends to 0 as  $k$  tends toward  $\infty$  through the integers.*

**Proof.** We have that

$$|1 - u^2| \leq 1 \quad (13)$$

for any  $u \in [-1, 1]$ . Using (13), we obtain that

$$\left| (1 - u^2)^{\nu-1/2} \right| \leq 1 \quad (14)$$

for any  $u \in [-1, 1]$ .

Moreover,

$$|e^{iuz}| \leq e^{|u||\text{Im} z|} \quad (15)$$

for any real number  $u$ , and any nonzero complex variable  $z$ , where  $\text{Im} z$  is the imaginary part of  $z$ . Using (15), we obtain that

$$|e^{iuz}| \leq e^{|\text{Im} z|} \quad (16)$$

for any  $u \in [-1, 1]$ , and any nonzero complex variable  $z$ .

Combining (5), (14), and (16) yields that

$$|J_{\nu+k}(z)| \leq \frac{2 e^{|\text{Im} z|} |z^\nu| |z|^k}{\sqrt{\pi} |2^\nu| 2^k |\Gamma(\nu + k + 1/2)|} \quad (17)$$

for any nonzero complex variable  $z$ , and any integer  $k$  such that  $k > |\nu| - 1$ .

We now fix any integer  $n$  such that  $n > |\nu| + \frac{1}{2}$ . Combining (17) and (10) yields that

$$|J_{\nu+k}(z)| \leq B_{\nu,n}(z) \frac{|z|^{k-n}}{2^{k-n} (k-n)!} \quad (18)$$

for any nonzero complex variable  $z$ , and any integer  $k$  such that  $k > n$ , where

$$B_{\nu,n}(z) = \frac{2 e^{|\text{Im} z|} |z^\nu| |z|^n}{\sqrt{\pi} |2^\nu| 2^n |\Gamma(\nu + n + 1/2)|} \quad (19)$$

for any nonzero complex variable  $z$ . Taking the limits of both sides of (18) as  $k$  tends toward  $\infty$  yields the present theorem.  $\square$

## 4 Identities

This section provides the principal results of the present note.

The following theorem provides an analogue for Bessel functions of what is known as the Christoffel-Darboux identity for orthonormal polynomials.

**Theorem 10** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\sum_{k=1}^{\infty} 2(\nu + k) J_{\nu+k}(w) J_{\nu+k}(z) = \frac{wz}{w-z} \left( J_{\nu+1}(w) J_{\nu}(z) - J_{\nu}(w) J_{\nu+1}(z) \right) \quad (20)$$

for any distinct nonzero complex variables  $w$  and  $z$ .

**Proof.** Using (8), we obtain that

$$\begin{aligned} & \sum_{k=1}^n \left( \frac{2(\nu + k)}{w} J_{\nu+k}(w) \right) J_{\nu+k}(z) - \sum_{k=1}^n J_{\nu+k}(w) \left( \frac{2(\nu + k)}{z} J_{\nu+k}(z) \right) \\ &= J_{\nu}(w) J_{\nu+1}(z) - J_{\nu+1}(w) J_{\nu}(z) + J_{\nu+n+1}(w) J_{\nu+n}(z) - J_{\nu+n}(w) J_{\nu+n+1}(z) \end{aligned} \quad (21)$$

for any distinct nonzero complex variables  $w$  and  $z$ , and any positive integer  $n$ ; dividing both the left- and right-hand sides of (21) by  $\frac{1}{w} - \frac{1}{z}$  and using Theorem 9 to take the limits as  $n$  tends toward  $\infty$  yields (20).  $\square$

**Remark 11** When  $\nu$  is a nonnegative integer, (20), (4), and (3) provide a basis for the filtering and interpolation of linear combinations of Bessel functions on  $(-\infty, \infty)$ , as originated for orthonormal polynomials in [3] and [12], and subsequently optimized.

The following theorem states the limit of (20) as  $w$  tends to  $z$ .

**Theorem 12** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\sum_{k=1}^{\infty} 2(\nu + k) (J_{\nu+k}(z))^2 = z^2 \left( J_{\nu}(z) \frac{d}{dz} J_{\nu+1}(z) - J_{\nu+1}(z) \frac{d}{dz} J_{\nu}(z) \right) \quad (22)$$

for any nonzero complex variable  $z$ .

**Proof.** Dividing both sides of (21) by  $\frac{1}{w} - \frac{1}{z}$ , we obtain that

$$\begin{aligned} & \sum_{k=1}^n 2(\nu + k) J_{\nu+k}(w) J_{\nu+k}(z) \\ &= \frac{wz}{w-z} \left( \left( J_{\nu+1}(w) - J_{\nu+1}(z) \right) J_{\nu}(z) - \left( J_{\nu}(w) - J_{\nu}(z) \right) J_{\nu+1}(z) \right) \\ & \quad + \frac{wz}{w-z} \left( \left( J_{\nu+n}(w) - J_{\nu+n}(z) \right) J_{\nu+n+1}(z) \right. \\ & \quad \left. - \left( J_{\nu+n+1}(w) - J_{\nu+n+1}(z) \right) J_{\nu+n}(z) \right) \end{aligned} \quad (23)$$

for any distinct nonzero complex variables  $w$  and  $z$ , and any positive integer  $n$ . Taking the limits of both sides of (23) as  $w$  tends to  $z$ , and then using (7), (8), and Theorem 9 to take the limits as  $n$  tends toward  $\infty$ , we obtain (22).  $\square$

The following theorem provides an alternative expression for the series in (22).

**Theorem 13** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$\sum_{k=1}^{\infty} 2(\nu + k) (J_{\nu+k}(z))^2 = z^2 (J_{\nu}(z))^2 + z^2 (J_{\nu+1}(z))^2 - (2\nu + 1) z J_{\nu}(z) J_{\nu+1}(z) \quad (24)$$

for any nonzero complex variable  $z$ .

**Proof.** Combining (22), (6), and (7) yields (24).  $\square$

The following theorem provides an expression for the sum of the terms with even indices in the series in (20).

**Theorem 14** *Suppose that  $\nu$  is a complex number.*

*Then,*

$$2(\nu + 1) \sum_{k=1}^{\infty} 2(\nu + 2k) J_{\nu+2k}(w) J_{\nu+2k}(z) = \frac{w^2 z^2}{w^2 - z^2} \left( J_{\nu+2}(w) J_{\nu}(z) - J_{\nu}(w) J_{\nu+2}(z) \right) \quad (25)$$

for any distinct nonzero complex variables  $w$  and  $z$ .

**Proof.** Using (9), we obtain that

$$\begin{aligned} & 2(\nu + 1) \left( \sum_{k=1}^n \left( \frac{2(\nu + 2k)}{w^2} J_{\nu+2k}(w) \right) J_{\nu+2k}(z) - \sum_{k=1}^n J_{\nu+2k}(w) \left( \frac{2(\nu + 2k)}{z^2} J_{\nu+2k}(z) \right) \right) \\ &= J_{\nu}(w) J_{\nu+2}(z) - J_{\nu+2}(w) J_{\nu}(z) + \frac{\nu + 1}{\nu + 2n + 1} \left( J_{\nu+2n+2}(w) J_{\nu+2n}(z) - J_{\nu+2n}(w) J_{\nu+2n+2}(z) \right) \end{aligned} \quad (26)$$

for any distinct nonzero complex variables  $w$  and  $z$ , and any positive integer  $n$ , provided that  $\nu$  is not an odd negative integer. When  $\nu$  is an odd negative integer, (26) holds for any distinct nonzero complex variables  $w$  and  $z$ , and any sufficiently large integer  $n$ , by continuity from the cases when  $\nu$  is not an odd negative integer. In all cases, dividing both sides of (26) by  $\frac{1}{w^2} - \frac{1}{z^2}$  and using Theorem 9 to take the limits as  $n$  tends toward  $\infty$  yields (25).  $\square$

**Remark 15** Together, (25) and (4) provide a basis for the filtering and interpolation of linear combinations of Bessel functions on  $(0, \infty)$ , as originated for orthonormal polynomials in [3] and [12], and subsequently optimized.

## 5 Generalizations

This section derives an analogue, for any family of functions satisfying a symmetric “three-term” recurrence relation, of what is known as the Christoffel-Darboux identity for orthonormal polynomials.

Suppose that  $g$  and  $\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots$  are complex-valued functions on a set  $S$ , and  $\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots$ , and  $\dots, d_{-2}, d_{-1}, d_0, d_1, d_2, \dots$  are complex numbers, such that

$$g(x) f_k(x) = c_{k-1} f_{k-1}(x) + d_k f_k(x) + c_k f_{k+1}(x) \quad (27)$$

for any  $x \in S$ , and any integer  $k$ .

Using (27), we obtain that

$$\begin{aligned} \sum_{k=m+1}^n (g(x) f_k(x)) f_k(y) - \sum_{k=m+1}^n f_k(x) (g(y) f_k(y)) &= c_m \left( f_m(x) f_{m+1}(y) - f_{m+1}(x) f_m(y) \right) \\ &+ c_n \left( f_{n+1}(x) f_n(y) - f_n(x) f_{n+1}(y) \right) \end{aligned} \quad (28)$$

for any  $x \in S$  and  $y \in S$ , and any integers  $m$  and  $n$  such that  $m < n$ .

Dividing both sides of (28) by  $g(x) - g(y)$ , we obtain that

$$\begin{aligned} \sum_{k=m+1}^n f_k(x) f_k(y) &= \frac{c_m}{g(x) - g(y)} \left( f_m(x) f_{m+1}(y) - f_{m+1}(x) f_m(y) \right) \\ &+ \frac{c_n}{g(x) - g(y)} \left( f_{n+1}(x) f_n(y) - f_n(x) f_{n+1}(y) \right) \end{aligned} \quad (29)$$

for any  $x \in S$  and  $y \in S$  such that  $g(x) \neq g(y)$ , and any integers  $m$  and  $n$  such that  $m < n$ ; (29) is analogous to the classical Christoffel-Darboux identity for orthonormal polynomials described, for example, in [5].

Section 4 implicitly applies (29) with  $m = 0$  for the following choices of functions:

$$g(x) = \frac{1}{x} \quad (30)$$

and

$$f_k(x) = \sqrt{2(\nu + k)} J_{\nu+k}(x), \quad (31)$$

as well as

$$g(x) = \frac{1}{x^2} \quad (32)$$

and

$$f_k(x) = \sqrt{2(\nu + 2k)} J_{\nu+2k}(x). \quad (33)$$

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